

# Cavity type problems ruled by infinity Laplacian operator

G. C. Ricarte\*, J. V. Silva† and R. Teymurazyan‡

## Abstract

We study a singularly perturbed problem related to infinity Laplacian operator with prescribed boundary values in a region. We prove that solutions are locally (uniformly) Lipschitz continuous, they grow as a linear function, are strongly non-degenerate and have porous level surfaces. Moreover, for some restricted cases we show the finiteness of the  $(n - 1)$ -dimensional Hausdorff measure of level sets. The analysis of the asymptotic limits is carried out as well.

**Keywords:** Infinity Laplacian, Lipschitz regularity, singularly perturbed problems, Hausdorff measure.

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## 1 Introduction

In this paper we study inhomogeneous singularly perturbed problems ruled by the *Infinity Laplacian*, which is defined as follows:

$$\Delta_{\infty} u := (Du)^T D^2 u Du = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j}.$$

More precisely, we study *weak* solutions to

$$\begin{cases} \Delta_{\infty} u^{\varepsilon}(x) &= \zeta_{\varepsilon}(x, u^{\varepsilon}) & \text{in } \Omega, \\ u^{\varepsilon}(x) &= \varphi^{\varepsilon}(x) & \text{on } \partial\Omega, \end{cases} \quad (E_{\varepsilon})$$

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\*GLEYDSON CHAVES RICARTE – Universidade Federal do Ceará, Av. Humberto Monte s/n, Campus do Pici - Bloco 914, 60455-760 Fortaleza, CE, Brazil. E-mail: [ricarte@mat.ufc.br](mailto:ricarte@mat.ufc.br)

†JOÃO VÍTOR DA SILVA – Universidad de Buenos Aires, Departamento de Matemática, Ciudad Universitaria-Pabellón I-(C1428EGA) - Buenos Aires, Argentina. E-mail: [jdasilva@dm.uba.ar](mailto:jdasilva@dm.uba.ar)

‡RAFAEL TEYMURAZYAN – Universidade de Coimbra, Departamento de Matemática, 3001-501 Coimbra, Portugal. E-mail: [rafayel@utexas.edu](mailto:rafayel@utexas.edu)

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain with a smooth boundary, and  $0 \leq \varphi^\varepsilon \in C(\overline{\Omega})$  with  $\|\varphi^\varepsilon\|_{L^\infty(\Omega)} \leq \mathcal{A}$ , for some constant  $\mathcal{A} > 0$ . The reaction term  $\zeta_\varepsilon$  represents the singular perturbation of the model. We are interested in singular behaviors of order  $O(\frac{1}{\varepsilon})$  along  $\varepsilon$ -level layers  $\{u_\varepsilon \sim \varepsilon\}$ , hence we consider (smooth) singular reaction terms  $\zeta_\varepsilon: \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying

$$0 \leq \zeta_\varepsilon(x, t) \leq \frac{\mathcal{B}}{\varepsilon} \chi_{(0, \varepsilon)}(t) + \mathcal{C}, \quad \forall (x, t) \in \Omega \times \mathbb{R}_+, \quad (1.1)$$

for some constants  $\mathcal{B}, \mathcal{C} \geq 0$ . Clearly  $\zeta_\varepsilon \equiv 0$  satisfies (1.1), therefore, to insure that the reaction term is genuinely singular, we will assume also that

$$\mathfrak{R} := \inf_{\Omega \times [a,b]} \varepsilon \zeta_\varepsilon(x, \varepsilon t) > 0, \quad (1.2)$$

for some  $0 \leq a < b$ , and  $\Re$  does not depend on  $\varepsilon$ . Heuristically, (1.2) says that the singular term behaves asymptotically as  $\sim \varepsilon^{-1} \chi_{(0, \varepsilon)}$  plus a nonnegative noise that stays uniformly bounded away from infinity. Singular reaction terms is built up as approximation of unity

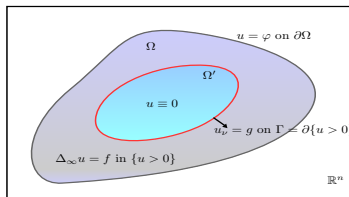
$$\zeta_\varepsilon(x, t) := \frac{1}{\varepsilon} \beta \left( \frac{t}{\varepsilon} \right) + g_\varepsilon(x), \quad (1.3)$$

are particular (simpler) cases covered by analysis to be developed herein (usually  $\beta$  is a nonnegative smooth real function with  $\text{supp } \beta = [0, 1]$ , and  $0 \leq c_0 \leq g_\varepsilon(x) \leq c_1 < \infty$ ). It is easy to check that the reaction term written in (1.3) satisfies (1.1) and (1.2).

We were motivated by the study of the following over-determined problem: given  $\Omega \subset \mathbb{R}^n$  a domain, functions  $0 \leq f, \varphi \in C(\overline{\Omega})$  and  $0 < g \in C(\overline{\Omega})$ , we would like to find a compact “hyper-surface”  $\Gamma := \partial\Omega' \subset \Omega$  such that the boundary value problem

$$\left\{ \begin{array}{lll} \Delta_\infty u(x) & = & f(x) \quad \text{in} \quad \Omega \setminus \Omega' \\ u(x) & = & \varphi(x) \quad \text{on} \quad \partial\Omega \\ u(x) & = & 0 \quad \text{on} \quad \Omega' \\ \frac{\partial u}{\partial \nu}(x) & = & g(x) \quad \text{on} \quad \Gamma \end{array} \right. \quad (1.4)$$

has a solution. Possible limiting functions coming from  $E_\varepsilon$  are natural choices to solve the above problem with  $\Gamma = \partial\{u > 0\}$  (the free boundary).



It is important to highlight that, unlike [2] and [11], we can not study  $(E_\varepsilon)$  as a limit of “variational solutions” of the corresponding inhomogeneous problem with  $p$ -Laplacian on the left hand side of  $(E_\varepsilon)$ , because several geometric properties and estimates deteriorate, when  $p \rightarrow +\infty$ , since they depend on  $p$  (see, for example, [4, 8, 12]). This indicates the importance of the non-variational approach.

Viscosity solutions of  $(E_\varepsilon)$  exhibit two “distinct” free boundaries: the first one is the set of critical points  $\mathcal{C}(u^\varepsilon) := \{x \in \Omega \mid \nabla u^\varepsilon(x) = 0\}$ , and the second one is the “physical” free boundary,  $\Gamma_\varepsilon = \{u^\varepsilon \sim \varepsilon\}$  ( $\varepsilon$ -level surfaces). We are able to control  $u^\varepsilon$  in terms of  $\text{dist}(x, \Gamma_\varepsilon)$  and see that these two free boundaries do not intersect.

A problem similar to  $(E_\varepsilon)$  for a fully nonlinear operators in the left hand side was studied in recent years. In fact, in [15] the authors study fully nonlinear uniformly elliptic equations of the form

$$F(x, D^2 u^\varepsilon) = \zeta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega,$$

where  $\zeta_\varepsilon \sim \frac{1}{\varepsilon} \chi_{(0, \varepsilon)}$ . They prove several analytical and geometrical properties of solutions (see also [14] for global regularity character and [12] for an approach with inhomogeneous forcing term). A non-variational setting of the problem was studied in [1], where the authors obtain existence and optimal regularity results for the class of fully nonlinear, anisotropic degenerate elliptic problems

$$|\nabla u^\varepsilon|^\gamma F(D^2 u^\varepsilon) = \zeta_\varepsilon(x, u^\varepsilon) \quad \text{in } \Omega, \quad \text{with } \gamma \geq 0.$$

These summarize current results for singularly perturbed non-variational problems.

We also remark that although regularity of infinity harmonic functions is well studied (see [6, 7, 16]), regularity results for the inhomogeneous problem  $\Delta_\infty u = f$  in  $\Omega$ , are relatively recent and less developed. In this direction it was shown in [9] that blow-ups are linear, if  $f \in C(\Omega) \cap L^\infty(\Omega)$ . As a consequence, viscosity solutions of the inhomogeneous problem are Lipschitz continuous and also everywhere differentiable, if  $f \in C^1(\Omega) \cap L^\infty(\Omega)$ . In [3] Lipschitz regularity was proved for a more general right hand side  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  provided  $f \in C(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ .

This paper is organized as follows: in section 2 we state some preliminary results, which we use later. In section 3 we prove optimal Lipschitz regularity (uniformly in  $\varepsilon$ ). In section 4 we prove geometric non-degeneracy properties of solutions. As a consequence a Harnack type inequality and porosity of level surfaces are proved. In section 5 we show that for some restricted cases the  $(n - 1)$ -dimensional Hausdorff measure of the free boundary is finite. The corresponding asymptotic limit as  $\varepsilon \rightarrow 0^+$  in  $(E_\varepsilon)$  is studied in the Section 6. We finish the paper analyzing the one-dimensional profile for the limiting free boundary problem in section 7.

## 2 Preliminary results

We start with the definition of the solution.

**Definition 2.1.** A function  $u \in C(\Omega)$  is called a viscosity sub-solution (super-solution) of

$$\Delta_\infty u = f(x, u(x)) \quad \text{in } \Omega,$$

if whenever  $\phi \in C^2(\Omega)$  and  $u - \phi$  has a local maximum (minimum) at  $x_0 \in \Omega$  there holds

$$\Delta_\infty \phi(x_0) \geq f(x_0, \phi(x_0)) \quad (\text{resp. } \leq f(x_0, \phi(x_0))).$$

A function  $u$  is a viscosity solution when it is a viscosity sub and super-solution at the same time.

As it was shown in [10], the Dirichlet problem

$$\begin{cases} \Delta_\infty v(x) = f(x) & \text{in } \Omega \\ v(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

has a unique viscosity solution for  $\Omega \subset \mathbb{R}^n$  bounded, provided  $g \in C(\partial\Omega)$  and either  $\sup_\Omega f < 0$  or  $\inf_\Omega f > 0$ . However, the uniqueness may fail, if  $f$  changes the sign (see the counter-example in [10, Appendix A]).

We recall a comparison principle result:

**Proposition 2.1 (Comparison Principle, see [3], [10]).** Let  $f \in C(\Omega)$  such that  $f > 0$ ,  $f < 0$  or  $f = 0$  in  $\Omega$ . If  $u, v \in C(\overline{\Omega})$  satisfy

$$\Delta_\infty u(x) \geq f(x) \geq \Delta_\infty v(x) \quad \text{in } \Omega, \quad (2.1)$$

then

$$\sup_\Omega (u - v) = \sup_{\partial\Omega} (u - v). \quad (2.2)$$

We construct solutions by Perron's method. We state the following theorem independently of the  $(E_\varepsilon)$  context, since it may be of independent interest. For the proof we refer to [15] (see also [1]).

**Theorem 2.1.** Let  $f \in C^{0,1}(\Omega \times [0, \infty))$  be a bounded real function. Suppose that there exist a viscosity sub-solution  $\underline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$  and super-solution  $\overline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$  to  $\Delta_\infty u = f(x, u)$  satisfying  $\underline{u} = \overline{u} = \varphi \in C(\partial\Omega)$ . Define the class of functions

$$\mathcal{S}_\varphi^f := \left\{ w \in C(\overline{\Omega}) \left| \begin{array}{l} w \text{ is a viscosity super-solution to} \\ \Delta_\infty u(x) = f(x, u) \text{ in } \Omega \text{ such that } \underline{u} \leq w \leq \overline{u} \\ \text{and } w = \varphi \text{ on } \partial\Omega \end{array} \right. \right\}.$$

Then,

$$u(x) := \inf_{w \in \mathcal{S}_\varphi^f} w(x), \quad \text{for } x \in \overline{\Omega} \quad (2.3)$$

is a continuous viscosity solution to  $\Delta_\infty u(x) = f(x, u)$  in  $\Omega$  with  $u = \varphi$  continuously on  $\partial\Omega$ .

Existence of the solution to problem  $(E_\varepsilon)$  follows by choosing  $\underline{u} := \underline{u}^\varepsilon$  and  $\overline{u} := \overline{u}^\varepsilon$  respectively as solutions to the following boundary value problems:

$$\begin{cases} \Delta_\infty \underline{u}^\varepsilon = \sup_{\Omega \times [0, \infty)} \zeta_\varepsilon & \text{in } \Omega \\ \underline{u}^\varepsilon = \varphi^\varepsilon & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta_\infty \overline{u}^\varepsilon = 0 & \text{in } \Omega \\ \overline{u}^\varepsilon = \varphi^\varepsilon & \text{on } \partial\Omega. \end{cases}$$

Then  $\underline{u}^\varepsilon \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$  and  $\overline{u}^\varepsilon \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$  (see [3], [9] and [10]) are respectively a viscosity sub and super-solutions of  $(E_\varepsilon)$ . We state this as a theorem:

**Theorem 2.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain and let  $\varphi^\varepsilon \in C(\partial\Omega)$  be a nonnegative boundary datum. Then for each fixed  $\varepsilon > 0$  there exists a (nonnegative) viscosity solution  $u^\varepsilon \in C(\overline{\Omega})$  to  $(E_\varepsilon)$ .*

As a consequence of Proposition 2.1, we get (uniform) boundness of any family of viscosity solutions.

**Lemma 2.1.** *Let  $u^\varepsilon$  be a viscosity solution to  $(E_\varepsilon)$ . Then there exists a constant  $C > 0$  independent of  $\varepsilon$  such that*

$$0 \leq u^\varepsilon(x) \leq C \quad \text{in } \Omega.$$

Next, we recall (see [14]) a Hopf's type lemma below for a future reference.

**Lemma 2.2.** *Let  $u$  be a viscosity solution to*

$$\begin{cases} \Delta_\infty u = f & \text{in } B_r(z) \\ u \geq 0 & \text{in } B_r(z). \end{cases}$$

*If for some  $x_0 \in \partial B_r(z)$ ,*

$$u(x_0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x_0) \leq \theta,$$

*where  $\nu$  is the inward normal vector at  $x_0$ , then there exists a universal constant  $C > 0$  such that*

$$u(z) \leq C\theta r.$$

**Notations.** We finish this section by introducing some notations which we shall use in the paper.

- ✓  $\Omega_\varepsilon := \{x \in \Omega \mid 0 \leq u^\varepsilon \leq \varepsilon\}$  means the  $\varepsilon$ -level region.
- ✓  $\Gamma_\varepsilon := \{x \in \Omega \mid u^\varepsilon = \varepsilon\}$  means the  $\varepsilon$ -level surfaces.
- ✓  $\mathfrak{P}(u_0, \Omega') := \{u_0 > 0\} \cap \Omega'$ .
- ✓  $\mathfrak{F}(u_0, \Omega') := \partial\{u_0 > 0\} \cap \Omega'$  shall mean the free boundary.
- ✓  $d_\varepsilon(x_0) := \text{dist}(x_0, \Omega_\varepsilon)$ .

- ✓  $\mathcal{N}_\delta(G) := \{x \in \mathbb{R}^n \mid \text{dist}(x, G) < \delta\}$  with  $G \subset \mathbb{R}^n$ .
- ✓  $\mathcal{L}^n$  denotes the  $n$ -dimensional Lebesgue measure.
- ✓  $\mathcal{H}^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure.
- ✓  $\Omega' \Subset \Omega$  means that  $\Omega' \subset \overline{\Omega'} \subset \Omega$ , and  $\overline{\Omega'}$  is compact ( $\Omega'$  is compactly contained in  $\Omega$ ).
- ✓  $\mathfrak{D}(u, B_r(x_0)) := \frac{\mathcal{L}^n(\{u>0\} \cap B_r(x_0))}{\mathcal{L}^n(B_r(x_0))}$  indicates the positive density.

**Remark 2.1.** *Throughout this paper universal constants are the ones depending only on physical parameters: dimension and structural properties of the problem, i. e. on  $n, \mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$ .*

### 3 Uniform Lipschitz regularity

In this section we prove that viscosity solutions to  $(E_\varepsilon)$  are (uniformly) locally Lipschitz continuous (which, in view of Theorem 4.1 below (see also Remark 6.1), is optimal).

**Theorem 3.1.** *Let  $u^\varepsilon$  be a viscosity solution to  $(E_\varepsilon)$ . For every  $\Omega' \Subset \Omega$ , there exists a positive constant  $C_0$ , independent of  $\varepsilon$ , such that*

$$\|\nabla u^\varepsilon\|_{L^\infty(\Omega')} \leq C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)).$$

*Proof.* At first we analyze the closed region  $\Omega_\varepsilon := \{0 \leq u^\varepsilon \leq \varepsilon\} \cap \Omega'$ . Let  $\varepsilon \ll \frac{1}{3}\text{dist}(\Omega', \partial\Omega)$ . We fix  $x_0 \in \Omega_\varepsilon$  and define  $v : B_1 \rightarrow \mathbb{R}$  by

$$v(y) := \frac{u^\varepsilon(x_0 + \varepsilon y)}{\varepsilon}.$$

Then one has

$$\Delta_\infty v = \varepsilon \zeta_\varepsilon(x_0 + \varepsilon y, \varepsilon v(y)) := f_\varepsilon(y) \quad \text{in } B_1$$

in the viscosity sense. From (1.1) we have that

$$0 \leq f_\varepsilon(y) \leq \mathcal{B} + \varepsilon \mathcal{C} \leq C_\star(\mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega)).$$

Since  $f_\varepsilon \in C^1$ , then  $v$  is locally differentiable and moreover (see Theorem 2 and Corollary 2 of [9]),

$$|\nabla v(0)| \leq 4 \sup_{B_1} v + \frac{1}{2} 4^{\frac{1}{3}} \|f_\varepsilon\|_{L^\infty(B_1)}^{\frac{1}{3}}. \quad (3.1)$$

Since

$$v(0) = \frac{u^\varepsilon(x_0)}{\varepsilon} \leq 1,$$

Lemma 2.1 and the Harnack inequality (see Theorem 7.1 of [3]) imply

$$\|v\|_{L^\infty(B_1)} \leq C(\mathcal{A}, \mathcal{B}, \mathcal{C}). \quad (3.2)$$

Combining (3.1) and (3.2), we get

$$|\nabla u^\varepsilon(x_0)| = |\nabla v(0)| \leq C_0, \quad (3.3)$$

for some  $C_0 = C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)) > 0$  independent of  $\varepsilon$ .

Now we turn our attention to the case of open region  $\{u^\varepsilon > \varepsilon\} \cap \Omega'$ . Let

$$\Gamma_\varepsilon := \{x \in \Omega' \mid u^\varepsilon(x) = \varepsilon\}.$$

For a fixed  $x_1 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$ , define  $r := \text{dist}(x_1, \Gamma_\varepsilon)$ . We define also a function  $v_r : B_1 \rightarrow \mathbb{R}$  by

$$v_r(y) := \frac{u^\varepsilon(x_1 + ry) - \varepsilon}{r},$$

and note that

$$\Delta_\infty v_r = r\zeta_\varepsilon(x_1 + ry, rv_r(y) + \varepsilon) := \mathbf{g}(y),$$

in the viscosity sense. The choice of  $r$  implies that  $u^\varepsilon(x_1 + ry) > \varepsilon$ , for every  $y \in B_1$ , thus, it follows from (1.1) that  $\mathbf{g}$  is smooth enough and bounded, independently of  $\varepsilon$ , i.e.,

$$\|\mathbf{g}\|_{L^\infty(B_1)} \leq K_0(\mathcal{B}, \mathcal{C}, \text{diam}(\Omega)).$$

Now let  $z_0 \in \Gamma_\varepsilon$  be such that  $r = |x_1 - z_0|$ . As in the previous case from (3.3) one has

$$|\nabla u^\varepsilon(z_0)| \leq C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)). \quad (3.4)$$

Moreover, for  $y_0 := \frac{z_0 - x_1}{|z_0 - x_1|} \in \partial B_1$  we have

$$v_r(y_0) = 0 \quad \text{and} \quad \frac{\partial v_r}{\partial \nu}(y_0) \leq |\nabla v_r(y_0)| \leq C_0.$$

Therefore, by the Lemma 2.2

$$v_r(0) \leq C(\mathcal{A}, \mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)),$$

and this finishes the proof.  $\square$

## 4 Further properties of solutions

In this section we prove several properties of solutions. In particular, we show that solutions grow as a linear function out of  $\varepsilon$ -level surfaces, inside  $\{u^\varepsilon > \varepsilon\}$ . This is an optimal estimate, when considered uniform in  $\varepsilon$ . The proof is based on building an appropriate barrier function. We consider degenerate elliptic equations of the form

$$\Delta_\infty u = \zeta(x, u) \quad \text{in } \mathbb{R}^n,$$

where the reaction term satisfies the non-degeneracy assumption:

$$\inf_{\mathbb{R}^n \times [a, b]} \zeta(x, t) > 0. \quad (4.1)$$

**Proposition 4.1 (Infinity Laplacian's Barrier).** *Let  $0 < a < b < 1$  be fixed. For  $\alpha$  and  $A_0$  positive numbers (to be chosen) a posteriori, there exists a radially symmetric function  $\Theta_L: \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying*

$$\begin{aligned} \checkmark \quad & \Theta_L \in W^{2,\infty}(\mathbb{R}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n), \\ \checkmark \quad & \Delta_\infty \Theta_L(x) \leq \zeta(x, \Theta_L(x)) \quad \text{in } \mathbb{R}^n, \end{aligned} \quad (4.2)$$

$\checkmark$  there exists a universal  $\kappa_0 > 0$  constant such that

$$\Theta_L(x) \geq 4\kappa_0 L \quad \text{for } |x| \geq 4L, \quad (4.3)$$

$$\text{where } L \geq L_0 := \sqrt{\frac{b-a}{A_0}}.$$

*Proof.* Define

$$\Theta_L(x) := \begin{cases} a & \text{for } 0 \leq |x| < L; \\ A_0(|x| - L)^2 + a & \text{for } L \leq |x| < L + L_0; \\ \psi(L) - \phi(L)|x|^{-\alpha} & \text{for } |x| \geq L + L_0. \end{cases} \quad (4.4)$$

where

$$\phi(L) = \frac{2}{\alpha} \sqrt{(b-a)A_0} (L + L_0)^{1+\alpha} \quad \text{and} \quad \psi(L) = b + \phi(L) (L + L_0)^{-\alpha}, \quad (4.5)$$

Clearly  $\Theta_L \in W^{2,\infty}(\mathbb{R}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)$ . Moreover, for  $0 \leq |x| < L$  the inequality (4.2) is true. In the region  $L \leq |x| < L + L_0$ , we have

$$D_i \Theta_L(x) = 2A_0 \frac{(|x| - L)}{|x|} x_i$$

and

$$D_{ij} \Theta_L(x) = 2A_0 \left[ \left( \frac{1}{|x|^2} - \frac{(|x| - L)}{|x|^3} \right) x_i \cdot x_j + \frac{(|x| - L)}{|x|} \delta_{ij} \right].$$

Therefore, we obtain

$$\begin{aligned} \Delta_\infty \Theta_L(x) &= \sum_{i,j=1}^n D_i \Theta_L \cdot D_j \Theta_L \cdot D_{ij} \Theta_L \\ &= 8A_0^3 \frac{(|x| - L)^2}{|x|^2} \sum_{i,j=1}^n \left[ \left( \frac{1}{|x|^2} - \frac{(|x| - L)}{|x|} \right) x_i^2 x_j^2 + \frac{|x| - L}{|x|} x_i \cdot x_j \delta_{ij} \right] \\ &= 8A_0^3 \frac{(|x| - L)^2}{|x|^2} \left[ \left( \frac{1}{|x|^2} - \frac{(|x| - L)}{|x|} \right) |x|^4 + \frac{(|x| - L)}{|x|} |x|^2 \right] \\ &= 8A_0^3 \frac{(|x| - L)^2}{|x|^2} |x|^2 = 8A_0^3 (|x| - L)^3 \leq 8A_0^3 L_0^3 \\ &= (2\sqrt{A_0(b-a)})^3. \end{aligned}$$



By construction

$$a \leq \Theta_L(x) \leq b$$

and so, for  $A_0$  sufficiently small, we get

$$\Delta_\infty \Theta_L(x) \leq \inf_{\mathbb{R}^n \times [a, b]} \zeta(x, t) \leq \zeta(x, \Theta_L(x)).$$

Now, let us turn our attention to the set  $|x| \geq L + L_0$ . Direct computation shows that

$$D_i \Theta_L(x) = \alpha \phi(L) \frac{x_i}{|x|^{\alpha+2}}$$

and

$$D_{ij} \Theta_L(x) = \alpha \phi(L) |x|^{-(\alpha+2)} \left( -\frac{(\alpha+2)}{|x|^2} x_i x_j + \delta_{ij} \right),$$

hence

$$\Delta_\infty \Theta_L(x) = -\alpha^3 \phi^3(L) (\alpha+1) \frac{1}{|x|^{3\alpha+4}}.$$

Finally, for  $\alpha > 0$  we get

$$\Delta_\infty \Theta_L(x) \leq 0 \leq \zeta(x, \Theta_L(x)).$$

Therefore,  $\Theta_L$  satisfies (4.2). Finally, by (4.5)

$$|x| \geq 4L \geq 2(L + L_0) = 2 \left( \frac{\phi(L)}{\psi(L) - b} \right)^{\frac{1}{\alpha}}$$

and hence

$$\Theta_L(x) = \psi(L) - \phi(L) |x|^{-\alpha} \geq \psi(L) - 2^{-\alpha} (\psi(L) - b) \geq C_\alpha \psi(L),$$

for  $\alpha > 1$ . Therefore,

$$\Theta_L(x) \geq 4\kappa_0 L,$$

where  $\kappa_0 > 0$  depends on  $n$  and  $(b - a)$ . □

## 4.1 Linear growth

In order to establish lower bounds on the growth speed of the solution to  $(E_\varepsilon)$  inside the set  $\{u^\varepsilon > \varepsilon\}$ , the strategy now is to consider appropriate scaling versions of the universal barrier  $\Theta_L$ .

**Theorem 4.1.** *Let  $u^\varepsilon$  be a solution of  $(E_\varepsilon)$ . There exists a universal constant  $c > 0$  such that for any  $x_0 \in \{u^\varepsilon > \varepsilon\}$  and  $0 < \varepsilon \leq d_\varepsilon(x_0) \ll 1$  one has*

$$u^\varepsilon(x_0) \geq c d_\varepsilon(x_0).$$

*Proof.* Without loss of generality we assume that  $x_0 = 0$ . Set  $\eta = \frac{d_\varepsilon(0)}{3}$  and define

$$\Theta_\varepsilon(x) := \varepsilon \Theta_{\frac{\eta}{4\varepsilon}}\left(\frac{x}{\varepsilon}\right).$$

Using (4.3) and (4.4) we verify that for  $4L_0\varepsilon \leq \eta$ ,

$$\Theta_\varepsilon(0) = a\varepsilon \quad \text{and} \quad \Theta_\varepsilon|_{\partial B_\eta} \geq \kappa_0\eta. \quad (4.6)$$

Now, we claim that there exists a  $z_0 \in \partial B_\eta$  such that

$$\Theta_\varepsilon(z_0) \leq u^\varepsilon(z_0). \quad (4.7)$$

In fact, if

$$\Theta_\varepsilon(x) > u^\varepsilon(x) \quad \text{in} \quad \partial B_\eta,$$

then the auxiliary function

$$v^\varepsilon := \min\{\Theta_\varepsilon, u^\varepsilon\}$$

would be a super-solution to  $(E_\varepsilon)$ , but  $v^\varepsilon$  is strictly below  $u^\varepsilon$ , which contradicts the minimality of  $u^\varepsilon$ . Therefore, by (4.6) and (4.7), we obtain

$$\kappa_0\eta \leq \Theta_\varepsilon(z_0) \leq u^\varepsilon(z_0) \leq \sup_{B_\eta} u^\varepsilon. \quad (4.8)$$

Furthermore,  $u^\varepsilon$  satisfies (in the viscosity sense)

$$c_0 \leq \Delta_\infty u^\varepsilon \leq c_1 \quad \text{in} \quad B_{3\eta}.$$

Hence, by Harnack inequality (see Theorem 7.1 of [3]), we get

$$\sup_{B_\eta} u^\varepsilon \leq 9u^\varepsilon(0) + 12\sigma \left( \left( \frac{3\eta}{2} \right)^4 c_1 \right)^{1/3}.$$

Thus, by (4.8)

$$u^\varepsilon(0) \geq \frac{1}{9} \left( \kappa_0 - C\eta^{1/3} \right) \eta.$$

Finally, by taking  $\eta > 0$  small enough we conclude

$$u^\varepsilon(0) \geq c\eta.$$

for some  $0 < c < 1$  (independent of  $\varepsilon$ ). □

As a consequence of the Lipschitz regularity, Theorem 3.1 and Theorem 4.1, we are able to completely control  $u^\varepsilon$  in terms of  $d_\varepsilon(x_0)$ .

**Corollary 4.1.** *For a sub-domain  $\Omega' \Subset \Omega$ , there exists  $C > 0$ , depending on universal parameters and  $\Omega'$ , such that for  $x_0 \in \mathfrak{P}(u^\varepsilon - \varepsilon, \Omega')$  and  $\varepsilon \leq d_\varepsilon(x)$ , there holds*

$$C^{-1}d_\varepsilon(x_0) \leq u^\varepsilon(x_0) \leq C d_\varepsilon(x_0).$$

*Proof.* The inequality from below is exactly the Theorem 4.1. Now take  $y_0 \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$ , such that  $|y_0 - x_0| = d_\varepsilon(x_0)$ . From Theorem 3.1,

$$u^\varepsilon(x_0) \leq C d_\varepsilon(x_0) + u^\varepsilon(y_0) \leq C d_\varepsilon(x_0),$$

and the corollary is proved.  $\square$

## 4.2 Strong non-degeneracy

Next we see that solutions are strongly non-degenerate close to  $\varepsilon$ -level sets. This means that the maximum of  $u^\varepsilon$  on the boundary of a ball  $B_r$  centered in  $\{u^\varepsilon > \varepsilon\}$  is of order  $r$ .

**Theorem 4.2.** *Let  $\Omega' \Subset \Omega$ . There exists a universal constant  $c > 0$  such that for  $x_0 \in \mathfrak{P}(u^\varepsilon - \varepsilon, \Omega')$ ,  $\varepsilon \leq \rho \ll 1$ , there holds*

$$c \rho < \sup_{B_\rho(x_0)} u^\varepsilon \leq c^{-1}(\rho + u^\varepsilon(x_0)).$$

*Proof.* By taking  $\Theta_\varepsilon(x) = \varepsilon \Theta_{\frac{\rho}{4\varepsilon}}(x)$  we have

$$u^\varepsilon(z) > \Theta_\varepsilon(z),$$

for some point  $z \in \partial B_\rho(x_0)$ . Note that

$$\kappa_0 \rho \leq \Theta_\varepsilon(z) < u^\varepsilon(z) \leq \sup_{B_\rho(x_0)} u^\varepsilon,$$

where  $\kappa_0$  is as in Proposition 4.1. The upper estimate is a direct consequence of the Lipschitz regularity.  $\square$

As a consequence we get a positive density result.

**Corollary 4.2.** *Let  $x_0 \in \{u^\varepsilon > \varepsilon\}$  and  $\varepsilon \leq \rho \ll 1$ . There exists a universal constant  $c_0 \in (0, 1)$  such that*

$$\mathfrak{D}(u^\varepsilon - \varepsilon, B_\rho(x_0)) \geq c_0.$$

*Proof.* As we saw in the previous theorem, there exists  $y_0 \in B_\rho(x_0)$  such that

$$u^\varepsilon(y_0) \geq c_0 \rho.$$

On the other hand, by Lipschitz regularity, for  $z \in B_{\kappa\rho}(y_0)$ , we have

$$u^\varepsilon(z) + C\kappa\rho \geq u^\varepsilon(y_0).$$

Thus, by using the estimates from above, we are able to choose  $\kappa > 0$  small enough in order to have

$$z \in B_{\kappa\rho}(y_0) \cap B_\rho(x_0) \quad \text{and} \quad u^\varepsilon(z) > \varepsilon.$$

So we conclude that there exists a portion of  $B_\rho(x_0)$  with volume of order  $\sim \rho^n$  within  $\{u^\varepsilon > \varepsilon\}$ . Therefore, we have a uniform positive density result for the solution of  $(E_\varepsilon)$ . More precisely,

$$\mathcal{L}^n(B_\rho(x_0) \cap \{u^\varepsilon > \varepsilon\}) \geq \mathcal{L}^n(B_\rho(x_0) \cap B_{\kappa\rho}(y_0)) = c_0 \mathcal{L}^n(B_\rho(x_0)),$$

for some constant  $c_0 > 0$  independent of  $\varepsilon$ .  $\square$

### 4.3 Harnack type inequality

For solutions of  $(E_\varepsilon)$  the Harnack inequality is valid for balls that touch the free boundary along the  $\varepsilon$ -layers, i.e.,  $\partial\{u^\varepsilon > \varepsilon\}$ .

**Theorem 4.3.** *Let  $u^\varepsilon$  be a solution of  $(E_\varepsilon)$ . Let also  $x_0 \in \{u^\varepsilon > \varepsilon\}$  and  $\varepsilon \leq d := d_\varepsilon(x_0)$ . Then,*

$$\sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) \leq C \inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x)$$

for a universal constant  $C > 0$  independent of  $\varepsilon$ .

*Proof.* Let  $z_1, z_2$  be extremal points for  $u^\varepsilon$  in  $\overline{B_{\frac{d}{2}}(x_0)}$ , i.e.,

$$\inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_1) \quad \text{and} \quad \sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_2).$$

Since  $d_\varepsilon(z_1) \geq \frac{d}{2}$ , by Corollary 4.1

$$u^\varepsilon(z_1) \geq C_1 d. \tag{4.9}$$

Moreover, by Theorem 4.2

$$u^\varepsilon(z_2) \leq C_2 \left( \frac{d}{2} + u^\varepsilon(x_0) \right). \tag{4.10}$$

Taking  $y \in \partial\{u^\varepsilon > \varepsilon\}$  such that  $d = |x_0 - y|$  and  $z \in \overline{B_d(y)} \cap \{u^\varepsilon > \varepsilon\}$ , we get from Corollary 4.1 and Theorem 4.2

$$u^\varepsilon(x_0) \leq \sup_{B_d(z)} u^\varepsilon \leq C_2(d + u^\varepsilon(z)) \leq C_3 d. \tag{4.11}$$

Combining (4.9), (4.10) and (4.11), we conclude

$$\sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) \leq C \inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x).$$

□

### 4.4 Porosity of the level surfaces

As a consequence of the growth rate and the non-degeneracy property, we get porosity of level sets.

**Definition 4.1.** *A set  $E \subset \mathbb{R}^n$  is called porous with porosity  $\delta > 0$ , if  $\exists R > 0$  such that*

$$\forall x \in E, \quad \forall r \in (0, R), \quad \exists y \in \mathbb{R}^n \quad \text{such that} \quad B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity  $\delta$  has Hausdorff dimension not exceeding  $n - c\delta^n$ , where  $c = c(n) > 0$  is a constant depending only on  $n$ . In particular, a porous set has Lebesgue measure zero (see, for example, [17]).

**Theorem 4.4.** *Let  $u^\varepsilon$  be a solution of  $(E_\varepsilon)$ . Then the level sets  $\partial\{u^\varepsilon > \varepsilon\}$  are porous with porosity constant independent of  $\varepsilon$ .*

*Proof.* Let  $R > 0$  and  $x_0 \in \Omega$  be such that  $\overline{B_{4R}(x_0)} \subset \Omega$ .

We aim to prove the set  $\mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0))$  is porous.

Let  $x \in \mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0))$ . For each  $r \in (0, R)$  we have  $\overline{B_r(x)} \subset B_{2R}(x_0) \subset \Omega$ . Let  $y \in \partial B_r(x)$  such that  $u^\varepsilon(y) = \sup_{\partial B_r(x)} u^\varepsilon$ . By non-degeneracy

$$u^\varepsilon(y) \geq cr, \quad (4.12)$$

where  $c > 0$  is a constant. On the other hand, we know that near the free boundary

$$u^\varepsilon(y) \leq Cd_\varepsilon(y), \quad (4.13)$$

where  $C > 0$  is a constant, and  $d_\varepsilon(y)$  is the distance of  $y$  from the set  $\overline{B_{2R}(x_0)} \cap \Gamma_\varepsilon$ . Now, from (4.12) and (4.13) we get

$$d_\varepsilon(y) \geq \delta r \quad (4.14)$$

for a positive constant  $\delta < 1$ .

Let now  $y^* \in [x, y]$  be such that  $|y - y^*| = \frac{\delta r}{2}$ , then it is not hard to see that

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x). \quad (4.15)$$

Indeed, for each  $z \in B_{\frac{\delta}{2}r}(y^*)$

$$|z - y| \leq |z - y^*| + |y - y^*| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r,$$

and

$$|z - x| \leq |z - y^*| + (|x - y| - |y^* - y|) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

and (4.15) follows.

Since by (4.14)  $B_{\delta r}(y) \subset B_{d_\varepsilon(y)}(y) \subset \{u^\varepsilon > \varepsilon\}$ , then

$$B_{\delta r}(y) \cap B_r(x) \subset \{u^\varepsilon > \varepsilon\},$$

which together with (4.15) provides

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u^\varepsilon > \varepsilon\} \subset B_r(x) \setminus \mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0)).$$

□

## 5 Hausdorff measure estimates

In this section we prove the finiteness of the  $(n - 1)$ -dimensional Hausdorff measure of level surfaces. For that we restrict ourselves to the case when the reaction term, which propagates up to the free boundary, is non-degenerate. Suppose that  $a = 0$  in (1.2) and for some  $b > 0$

$$\mathfrak{R}_0 := \inf_{\Omega \times [0, b]} \varepsilon \zeta_\varepsilon(x, \varepsilon t) > 0. \quad (5.1)$$

**Definition 5.1 (Asymptotic Concavity Property).** *We say that an operator  $F : \Omega \times \text{Sym}(n) \rightarrow \mathbb{R}$  is asymptotically concave, if there exists*

$$\mathfrak{A} \in \mathcal{A}_{\lambda, \Lambda} := \left\{ A \in \text{Sym}(n) \mid \lambda \|\xi\|^2 \leq \sum_{i,j=1}^n A_{ij} \xi_i \xi_j \leq \Lambda \|\xi\|^2, \forall \xi \in \mathbb{R}^n \right\}$$

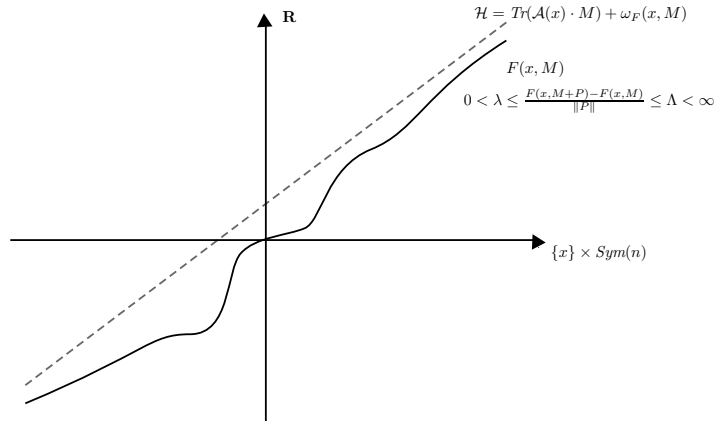
and a continuous function  $\omega_F : \Omega \times \text{Sym}(n) \rightarrow \mathbb{R}$  such that

$$F(x, M) \leq \text{Tr}(\mathfrak{A}(x) \cdot M) + \omega_F(x, M), \forall (x, M) \in \Omega \times \text{Sym}(n), \quad (\text{ACP})$$

with

$$\lim_{\|M\| \rightarrow \infty} |\omega_F(x, M)| := \mathcal{K} < \infty, \quad \forall x \in \Omega. \quad (5.2)$$

**Remark 5.1.** *The (ACP) condition is weaker than concavity assumption. Geometrically, it means that for each  $x \in \Omega$  fixed, there exists a hyperplane which decomposes  $\mathbb{R} \times \text{Sym}(n)$  in two semi-spaces such that the graph of  $F(x, \cdot)$  is always below this hyperplane. Moreover, by assuming  $F(x, 0) = 0$ , the assumption (5.2) means that the distance from the hyperplane to the graph of  $F$  goes to infinity for matrices with big enough norms (see [1] and [13]).*



**Definition 5.2.** Let  $v$  be the solution of  $(E_\varepsilon)$ . We write  $v \in \mathcal{S}(F, G, H)$ , if

$$\Delta_\infty v \leq G(|Dv|)F(x, D^2v) + H(x, |Dv|),$$

where

- ✓  $F : \Omega \times \text{Sym}(n) \rightarrow \mathbb{R}$  is a fully nonlinear uniformly elliptic operator with  $F(x, 0) = 0$ ;
- ✓  $G : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a non-negative continuous function and injective;
- ✓  $H : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is a bounded continuous function.

**Example 1 ( $\varphi$ -Laplacian operator).** The  $\varphi$ -Laplacian operator in Orlicz-Sobolev spaces can be defined as

$$\Delta_\varphi u = \frac{\varphi(|\nabla u|)}{|\nabla u|} \left[ \Delta u + \left\{ \frac{\varphi'(|\nabla u|)|\nabla u|}{\varphi(|\nabla u|)} - 1 \right\} \frac{\Delta_\infty u}{|\nabla u|^2} \right].$$

for an appropriate increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$  satisfying the generalized Ladyzhenskaya-Ural'tseva condition:

$$0 < g_0 \leq \frac{\varphi'(t)t}{\varphi(t)} \leq g_1, \quad \text{if } t > 0,$$

where  $g_0$  and  $g_1$  are constants. Therefore, for a  $\varphi$ -harmonic function one has (where  $\nabla u \neq 0$ )

$$\Delta_\infty u \leq \frac{\varphi(|\nabla u|)|\nabla u|^2}{\varphi'(|\nabla u|)|\nabla u| - \varphi(|\nabla u|)} \Delta u.$$

**Example 2 (Convex functions).** For convex functions we have following relation

$$\Delta_\infty u = \langle D^2 u Du, Du \rangle \leq |\nabla u|^2 \Delta u,$$

since  $\|D^2 u\|$  is controlled by  $\Delta u$ .

The proof of the following proposition is similar to the corresponding result from [1]. We sketch it here for reader's convenience.

**Proposition 5.1.** For the every fixed  $\Omega' \Subset \Omega$ ,  $\rho < \text{dist}(\Omega', \partial\Omega)$  and  $C \gg 1$ , there exists a universal  $\varepsilon_0$  such that

$$\int_{B_\rho(x_\varepsilon)} [\zeta_\varepsilon(x, u^\varepsilon(x)) - C] dx \geq 0, \quad (5.3)$$

for any  $x_\varepsilon \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$  whenever  $\varepsilon \leq \varepsilon_0$ .

*Proof.* If (5.3) is not true, then there are  $C_0 > 0$  and  $\rho < \text{dist}(\Omega', \partial\Omega)$  such that

$$\int_{B_\rho(x_k)} (\zeta_{\varepsilon_k}(x, u^{\varepsilon_k}) - C_0) dx < 0,$$

for points  $x_{\varepsilon_k} \in \mathfrak{F}(u^{\varepsilon_k} - \varepsilon_k, \Omega')$  and a sequence  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . Define

$$v_k(y) := \frac{bu^{\varepsilon_k}(x_{\varepsilon_k} + \varepsilon_k y)}{\varepsilon_k}.$$

Then

$$\int_{B_{\rho/\varepsilon_k}} ((\varepsilon_k b^{-1}) \zeta_{\varepsilon_k}(x_{\varepsilon_k} + \varepsilon_k y, \varepsilon_k b^{-1} v_k) - C_0 \varepsilon_k b^{-1}) dx < 0. \quad (5.4)$$

Note that

$$\|\Delta_\infty v_k\|_{L^\infty(B_{\rho/\varepsilon_k})} \leq \frac{\mathcal{B} + \mathcal{C}}{b},$$

independent of  $\varepsilon$ .

By the regularity of  $v_k$  one has (up to a subsequence) that

$$v_\infty := \lim_{k \rightarrow \infty} v_k,$$

in the  $C_{\text{loc}}^{0,\alpha}$  topology. Combining (5.1) and (5.4), we deduce that

$$\text{either } v_\infty \equiv 0, \quad \text{or else } v_\infty \geq b, \text{ everywhere in } \mathbb{R}^n.$$

The first case is not possible since  $v_\infty(0) = b > 0$ . If  $v_\infty \geq b$ , we have that 0 is a minimum point, which leads to a contradiction, since by non-degeneracy

$$0 = |\nabla v_\infty(0)| = |\nabla u^{\varepsilon_k}(0)| + o(1) \geq c > 0.$$

□

Thus, combining the **(ACP)** condition and the Proposition 5.1, we obtain:

**Lemma 5.1.** *Let  $u^\varepsilon \in \mathcal{S}(F, G, H)$  with  $F$  being asymptotically concave and let  $x_\varepsilon \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$ . Then*

$$\int_{B_\rho(x_\varepsilon)} A_{ij} u_{ij}^\varepsilon dx \geq 0. \quad (5.5)$$

*Proof.* Note that

$$F(x, D^2 u^\varepsilon) \geq [\zeta_\varepsilon(x, u^\varepsilon) - H(x, |\nabla u^\varepsilon|)] G(|\nabla u^\varepsilon|)^{-1}$$

in  $\{u^\varepsilon > \varepsilon\} \cap \Omega'$ , for any  $\Omega' \Subset \Omega$ . Hence, by Lipschitz regularity and properties of  $G$  and  $H$ , one has

$$F(x, D^2 u^\varepsilon) \geq [\zeta_\varepsilon(x, u^\varepsilon) - C_H] G(C)^{-1}.$$

Therefore, by **(ACP)** condition

$$\begin{aligned} \int_{B_\rho(x_\varepsilon)} A_{ij} u_{ij}^\varepsilon dx &\geq \int_{B_\rho(x_\varepsilon)} [(\zeta_\varepsilon(x, u^\varepsilon) - C_H) G(C)^{-1} - \mathcal{K}] dx \\ &\geq G(C)^{-1} \int_{B_\rho(x_\varepsilon)} [\zeta_\varepsilon(u^\varepsilon) - (C_H + G(C)\mathcal{K})] dx, \end{aligned}$$

where  $C > 0$  comes from the universal control on the Lipschitz norm in  $B_\rho(x_\varepsilon)$ . Combining the estimate above and the Proposition 5.1, we obtain (5.5). □



Lemma 5.1 plays a crucial role in the study of regularity of level surfaces, since it leads to the following result (see Theorem 5.6 in [1]):

**Theorem 5.1.** *Let  $\Omega' \Subset \Omega$  and  $u^\varepsilon \in \mathcal{S}(F, G, H)$  with  $F$  being asymptotically concave. There exists a  $C > 0$  constant depending on  $\Omega'$  such that*

$$\mathcal{H}^{n-1}(\mathfrak{P}(u^\varepsilon - C_1\varepsilon, B_\rho(x_\varepsilon))) \leq C\rho^{n-1}, \quad (5.6)$$

for some  $C_1 > 1$  and for all  $x_\varepsilon \in \mathfrak{F}(u^\varepsilon - C_1\varepsilon, \Omega')$ , provided  $d_\varepsilon(x_\varepsilon) < \text{dist}(\Omega', \partial\Omega)$  and  $C_1\varepsilon \leq \rho$ .

## 6 The limiting problem

As a consequence of Theorem 3.1 and Lemma 2.1 we obtain the following result:

**Theorem 6.1.** *If  $\{u^\varepsilon\}_{\varepsilon>0}$  is a solution to  $(E_\varepsilon)$ , then for any sequence  $\varepsilon_k \rightarrow 0^+$  there exist a subsequence  $\varepsilon_{k_j} \rightarrow 0^+$  and  $u_0 \in C_{\text{loc}}^{0,1}(\Omega)$  such that*

- (1)  $u^{\varepsilon_{k_j}} \rightarrow u_0$  locally uniformly in  $\Omega$ ;
- (2)  $0 \leq u_0(x) \leq K_0$  in  $\overline{\Omega}$  for some constant  $K_0$  independent of  $\varepsilon$ ;
- (3)  $\Delta_\infty u_0(x) = g(x)$  in  $\Omega \setminus \mathfrak{F}(u_0, \Omega')$ , with  $g$  being a bounded and nonnegative continuous function.

**Remark 6.1.** *It follows from (3) (using the corresponding regularity result from [9]) that  $u_0$  is locally differentiable in  $\mathfrak{P}(u_0, \Omega')$ . However, that property deteriorates as  $\text{dist}(\partial\Omega', \partial\{u_0 > 0\}) \rightarrow 0$ . On the other hand, the gradient remains controlled even when  $\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \rightarrow 0$ .*

Hereafter we will use the following definition when referring to  $u_0$ :

$$u_0(x) := \lim_{j \rightarrow \infty} u^{\varepsilon_j}(x).$$

**Theorem 6.2.** *Let  $\Omega' \Subset \Omega$ . Fix  $x_0 \in \mathfrak{P}(u_0, \Omega')$  such that  $\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq \text{dist}(\Omega', \partial\Omega)$ . Then there exists a constant  $C > 0$  independent of  $\varepsilon$  such that*

$$C^{-1}\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq u_0(x_0) \leq C\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')). \quad (6.1)$$

*Proof.* From Corollary 4.1 we know that there exists  $y_\varepsilon \in \Omega_\varepsilon$  such that

$$d_\varepsilon(x) = |x - y_\varepsilon| \text{ and } u^\varepsilon(x) \geq c d_\varepsilon(x) = c|x - y_\varepsilon|,$$

for some constant  $c > 0$  independent of  $\varepsilon$ . Passing to a subsequence, if necessary, we get for  $y_\varepsilon \rightarrow y_0 \in \mathfrak{F}(u_0, \Omega')$

$$u_0(x) \geq c|x_0 - y_0| \geq c\text{dist}(x, \mathfrak{F}(u_0, \Omega')).$$

Finally, the upper bound is a consequence of the local Lipschitz estimate for  $u_0$ .  $\square$

The next theorem is an immediate consequence of Theorem 4.2 as  $\varepsilon \rightarrow 0^+$ .

**Theorem 6.3.** *Let  $\Omega' \Subset \Omega$ . For any  $x_0 \in \mathfrak{P}(u_0, \Omega')$  such that  $\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq \text{dist}(\Omega', \partial\Omega)$ , there exist constants  $C_0 > 0$  and  $r_0 > 0$  independent of  $\varepsilon$ , such that*

$$C_0^{-1}r \leq \sup_{B_r(x_0)} u_0 \leq C_0(r + u_0(x_0))$$

provided  $r \leq r_0$ .

The following result shows that, in Hausdorff distance,  $\Omega_\varepsilon$  converges to  $\mathfrak{P}(u_0, \Omega')$  as  $\varepsilon \rightarrow 0^+$ .

**Theorem 6.4.** *Let  $\Omega' \Subset \Omega$ . Then for a  $C_1 > 1$ , the following inclusions hold:*

$$\mathfrak{P}(u_0, \Omega') \subset \mathcal{N}_\delta(\{u^{\varepsilon_j} > C_1\varepsilon_j\}) \cap \Omega' \text{ and } \{u^{\varepsilon_j} > C_1\varepsilon_j\} \cap \Omega' \subset \mathcal{N}_\delta(\{u_0 > 0\}) \cap \Omega',$$

provided  $\varepsilon_j \leq \delta \ll 1$ .

*Proof.* We prove the first inclusion (the other one can be obtained in a similar way). Suppose that it is not true. Then there exists a  $\delta_0 > 0$  such that for every  $\varepsilon_j \rightarrow 0$  and  $\forall x_j \in \mathfrak{P}(u_0, \Omega')$

$$\text{dist}(x_j, \{u^{\varepsilon_j} > C_1\varepsilon_j\}) > \delta_0. \quad (6.2)$$

For some  $y \in \overline{B_{\frac{\delta_0}{2}}(x_j)} \cap \{u^{\varepsilon_j} > C_1\varepsilon_j\}$  we have from Theorem 6.3

$$u^{\varepsilon_j}(y) = \sup_{B_{\frac{\delta_0}{2}}(x_j)} u^{\varepsilon_j}(x_j) \geq \frac{1}{2} \sup_{B_{\frac{\delta_0}{2}}(x_j)} u_0(x_j) \geq c\delta_0 \geq C_1\varepsilon_j,$$

which contradicts (6.2).  $\square$

**Theorem 6.5.** *Given  $\Omega' \Subset \Omega$ , there exist constants  $C > 0$  and  $\rho_0 > 0$ , depending only on  $\Omega'$  and universal parameters, such that for any  $x_0 \in \mathfrak{F}(u_0, \Omega')$  there holds*

$$C^{-1}\rho \leq \int_{\partial B_\rho(x_0)} u_0(x) d\mathcal{H}^{n-1} \leq C\rho. \quad (6.3)$$

provided  $\rho \leq \rho_0$ .

*Proof.* The upper bound follows from the Lipschitz regularity of  $u_0$ . The lower bound is a consequence of the nondegeneracy.  $\square$

**Remark 6.2.** *Repeating the steps of the proof of Theorem 4.3 one can show that the Harnack inequality is true for  $u_0$  in touching balls. Furthermore, as a consequence of the non-degeneracy and the growth rate, one can prove (as it was done in Theorem 4.4) that the free boundary  $\mathfrak{F}(u_0)$  is a porous set.*

Next, we prove several geometric-measure properties for  $\mathfrak{F}(u_0)$ . The ultimate goal is to prove the local finiteness of the  $(n-1)$ -dimensional Hausdorff measure of the limiting level surface.

First we see that the set  $\{u_0 > 0\}$  has uniform density along  $\mathfrak{F}(u_0)$ .

**Theorem 6.6.** *Let  $\Omega' \Subset \Omega$ . There exists a constants  $c_0 > 0$  such that for any  $x_0 \in \mathfrak{F}(u_0, \Omega')$  there holds*

$$\mathfrak{D}(u_0, B_\rho(x_0)) \geq c_0 \quad (6.4)$$

*provided  $\rho \ll 1$ . In particular,  $\mathcal{L}^n(\mathfrak{F}(u_0)) = 0$ .*

*Proof.* The estimate (6.4) follows as in the proof of Corollary 4.2. We conclude the resuly by using Lebesgue differentiation theorem and a covering argument (Besicovitch-Vitali type theorem, see [5]).  $\square$

**Theorem 6.7.** *Let  $\Omega' \Subset \Omega$ . There exists a constant  $C > 0$ , depending only on  $\Omega'$  and universal parameters such that, for any  $x_0 \in \mathfrak{F}(u_0, \Omega')$ , there holds*

$$\mathcal{H}^{n-1}(\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)) \leq C\rho^{n-1}.$$

*Proof.* From Theorem 6.4, for  $j \gg 1$  one has

$$[\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0)] \subset [\mathcal{N}_{4\delta}(\partial\{u^{\varepsilon_j} > C_1\varepsilon_j\}) \cap B_{2\rho}(x_0)].$$

Assuming  $\varepsilon_j \leq \delta \leq \rho \ll \text{dist}(\Omega', \partial\Omega)$ , the hypotheses of Theorem 5.1 are fulfilled, implying the following estimate for the  $\delta$ -neighborhood,

$$\mathcal{L}^n(\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0)) \leq C\delta\rho^{n-1}.$$

Now, let  $\{B_j\}_{j \in \mathbb{N}}$  be a covering of  $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$  by balls with radii  $\delta > 0$  and centered at free boundary points on  $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$ . Then

$$\bigcup_j B_j \subset \mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_{\rho+\delta}(x_0).$$

Therefore, there exists a constant  $\overline{C} > 0$  with universal dependence such that

$$\begin{aligned} \mathcal{H}_\delta^{n-1}(\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)) &\leq \overline{C} \sum_j \mathcal{L}^{n-1}(\partial B_j) \\ &= n \frac{\overline{C}}{\delta} \mathcal{L}^n(B_j) \\ &\leq n \frac{\overline{C}}{\delta} \mathcal{L}^n(\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_{\rho+\delta}(x_0)) \\ &\leq C(n)(\rho + \delta)^{n-1} \\ &= C(n)\rho^{n-1} + o(\delta). \end{aligned}$$

Letting  $\delta \rightarrow 0^+$  we finish the proof.  $\square$

As an immediate consequence of Theorem 6.7 we conclude that  $\mathfrak{F}(u_0)$  has locally finite perimeter. Moreover, the reduced free boundary  $\mathfrak{F}^*(u_0) := \partial_{\text{red}}\{u_0 > 0\}$  has a total  $\mathcal{H}^{n-1}$  measure in the sense that  $\mathcal{H}^{n-1}(\mathfrak{F}(u_0) \setminus \mathfrak{F}^*(u_0)) = 0$  (Theorem 6.7 in [1]). In particular, the free boundary has an outward vector for  $\mathcal{H}^{n-1}$  almost everywhere in  $\mathfrak{F}^*(u_0)$ .

## 7 Final comments

We finish the paper by analysing the one-dimensional profile representing the corresponding free boundary condition. Let

$$u_{xx}^\varepsilon (u_x^\varepsilon)^2 = \zeta_\varepsilon(u^\varepsilon) \quad \text{in} \quad (-1, 1), \quad (7.1)$$

where  $\zeta_\varepsilon$  given by

$$\zeta_\varepsilon(s) = \frac{1}{\varepsilon} \zeta\left(\frac{s}{\varepsilon}\right)$$

is a high energy activation potential, i.e., a non-negative smooth function supported in  $[0, \varepsilon]$ . The limiting configuration satisfies (in the viscosity sense)

$$\Delta_\infty u_0 = 0 \quad \text{in} \quad \{u_0 > 0\} \cap (-1, 1).$$

Multiplying (7.1) by  $u_x^\varepsilon$  we get

$$u_{xx}^\varepsilon (u_x^\varepsilon)^3 = \zeta_\varepsilon(u^\varepsilon) \cdot u_x^\varepsilon = \frac{d}{dx} \Xi_\varepsilon(u^\varepsilon), \quad (7.2)$$

where

$$\Xi_\varepsilon(t) = \int_0^{\frac{t}{\varepsilon}} \zeta(s) ds \rightarrow \left( \int \zeta(s) ds \right) \chi_{\{t > 0\}}$$

as  $\varepsilon \rightarrow 0^+$ , i.e.,

$$\Xi_\varepsilon(u^\varepsilon) \rightarrow \int \zeta(s) ds, \quad \text{as} \quad \varepsilon \rightarrow 0^+$$

provided  $u_0(x) > 0$ . Using change of variable

$$u_x^\varepsilon(x) = w,$$

we re-write

$$\int \frac{d}{dx} \Xi_\varepsilon(u^\varepsilon) = \int (u^\varepsilon)_x^3 u_{xx}^\varepsilon dx = \int w^3 dw.$$

Hence, by computing the anti-derivatives at (7.2) and letting  $\varepsilon \rightarrow 0^+$  we obtain the following characterization for limiting condition

$$|u'_0| = \sqrt[4]{4 \int \zeta(s) ds} \quad \text{on} \quad \partial\{u_0 > 0\}.$$

Therefore, the corresponding one-dimensional limiting free boundary problem is given by

$$\begin{cases} \Delta_\infty u_0 &= 0 & \text{in} & \{u_0 > 0\} \cap (-1, 1), \\ u_0 &= 0 & \text{in} & \partial\{u_0 > 0\}, \\ |u'_0| &= \sqrt[4]{4 \int \zeta(s) ds} & \text{on} & \partial\{u_0 > 0\}. \end{cases}$$

Furthermore, if for some direction  $x_i$  we have

$$u_{x_i x_i}^\varepsilon (u_{x_i}^\varepsilon)^2 \leq \zeta_\varepsilon(u^\varepsilon) \quad \text{in} \quad \Omega,$$

then by repeating the previous argument (since  $u^\varepsilon$  is increasing in direction  $x_i$ ), we conclude

$$\left| \frac{\partial u_0}{\partial x_i} \right| \leq \sqrt[4]{4 \int \zeta(s) ds} \quad \text{on } \partial\{u_0 > 0\}$$

in every regular point of the free boundary.

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